

A uniform contraction principle for bounded Apollonian embeddings.

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Abstract

Let $\widehat{H} = H \cup \{\infty\}$ denote the standard one-point completion of a real Hilbert space H . Given any non-trivial proper sub-set $U \subset \widehat{H}$ one may define the so-called ‘Apollonian’ metric d_U on U . When $U \subset V \subset \widehat{H}$ are nested proper subsets we show that their associated Apollonian metrics satisfy the following uniform contraction principle: Let $\Delta = \text{diam}_V(U) \in [0, +\infty]$ be the diameter of the smaller subsets with respect to the large. Then for every $x, y \in U$ we have

$$d_V(x, y) \leq \tanh \frac{\Delta}{4} d_U(x, y).$$

In dimension one, this contraction principle was established by Birkhoff [Bir57] for the Hilbert metric of finite segments on \mathbb{RP}^1 . In dimension two it was shown by Dubois in [Dub09] for subsets of the Riemann sphere $\widehat{\mathbb{C}} \sim \widehat{\mathbb{R}^2}$. It is new in the generality stated here.

1 Introduction and results

There are striking similarities between the projective group for the real or complex projective lines and the conformal group of the one-point completion of a real Hilbert space of dimension at least 3. In the first case, the group consists of Möbius maps of the form $z \mapsto \frac{az+b}{cz+d}$ and in the second it is generated by linear isometries, homotheties and the inversion, corresponding to Möbius transformations supplemented with a complex conjugation. In both cases one needs at least 4 points to define a group invariant quantity, i.e. the cross-ratio. Fixing a subset U whose complement contains at least 2 points, the logarithm of cross-ratios may then be used to construct a (semi-)metric on U . On the interval $I = (-1, 1)$, there is a unique (up to a constant) distance invariant under Möbius transformations preserving I . This is precisely the restriction of the Poincaré metric $2|dz|(1 - |z|^2)^{-1}$ on the unit disk in the complex plane. In the case of Hilbert spaces of higher dimensions one may derive the so-called ‘Apollonian metric’ (see below). This latter metric was first introduced for $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ by Barbilian [Bar34] and later rediscovered by Beardon [Bea98].

From a dynamical point of view it is of interest to know how a subset U metrically embed into a larger subset V with respect to the associated metrics d_U and d_V (see below for more precise

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statements). It is straight-forward from definitions that the injection $i : (U, d_U) \hookrightarrow (V, d_V)$ is non-expanding. More interesting, however, is that it verifies a very general uniform contraction principle (UCP): If $\Delta = \text{diam}_V(U) < +\infty$, i.e. the embedding of U has bounded diameter in the larger domain V , then the injection is a strict contraction with a Lipschitz constant bounded by $\tanh \frac{\Delta}{4}$. This is the same formula which appears in Birkhoff's work on the Hilbert metric [Bir57]. We give below the (surprisingly simple) proof of the UCP for the general case and in section 2 some simple dynamical systems applications.

There is no particular reasons for sticking to finite dimension, so in the following let H be any real Hilbert space. We write $\langle \cdot, \cdot \rangle$ for the scalar product and $\| \cdot \|$ for the norm on H . Let $\hat{H} = H \cup \{\infty\}$ be a one point completion of H in which the open sets containing ∞ are of the form $\{\infty\} \cup F^c$ with F a bounded closed set. With this convention \hat{H} is compact iff H is finite dimensional. The space (\hat{H}, \hat{d}) is a complete metric space of diameter one with respect to the metric:

$$\hat{d}(x, y) = \frac{\|x - y\|}{\sqrt{1 + \langle x, x \rangle} \sqrt{1 + \langle y, y \rangle}}, \quad \hat{d}(\infty, y) = \frac{1}{\sqrt{1 + \langle y, y \rangle}}. \quad (1.1)$$

Definition 1.1 *Given four points $\{x_1, x_2, u_1, u_2\} \in \hat{H}$ such that $\{x_1, x_2\}$ and $\{u_1, u_2\}$ are disjoint we define their cross-ratio to be :*

$$[x_1, x_2; u_1, u_2] \equiv \frac{\|x_2 - u_1\| \|x_1 - u_2\|}{\|x_1 - u_1\| \|x_2 - u_2\|}. \quad (1.2)$$

Here, $\| \cdot \|$ denotes the Hilbert norm in H and we adapt usual conventions for dealing with the point at ∞ . When $U \subset \hat{H}$ is a proper subset (by proper we mean that U and U^c are both non-empty) one defines the Apollonian (semi,pseudo-)distance between points $x_1, x_2 \in U$:

$$d_U(x_1, x_2) = \sup_{u_1, u_2 \in U^c} \log[x_1, x_2; u_1, u_2] \in [0, +\infty] \quad (1.3)$$

We denote by $GM(\hat{H})$ the general conformal group which acts continuously upon (\hat{H}, \hat{d}) and is generated by the set of isometries, homotheties (both fixing ∞) and the inversion (which exchanges the origin and ∞):

$$I(x) = \frac{x}{\langle x, x \rangle}, \quad I(0) = \infty \quad \text{and} \quad I(\infty) = 0. \quad (1.4)$$

When $\dim H \geq 3$ the Liouville theorem (see e.g. [Nev60]) shows that any conformal map is in $GM(\hat{H})$. In dimension 1 or 2, it is the Möbius group (supplemented with complex conjugation in the 2 dimensional case). That d_U is $GM(\hat{H})$ invariant is trivial for isometries and homotheties and in the case of inversions it follows easily from the formula $\|I(x) - I(y)\| = \frac{\|x - y\|}{\|x\| \|y\|}$ (with some care taken with respect to the point at infinity). From the cross-ratio identity $[x, z; u, v] = [x, y; u, v] [y, z; u, v]$ and taking sup in the right order one also sees that d_U verifies the triangular inequality. When U^c has non-empty interior d_U is a genuine metric, but in the general case it need not distinguish points. We refer to e.g. [Bea98, Chapter 3] and [Has04] for further details on the geometry of this metric. Our main result is the following:

Theorem 1.2 [Main Theorem] *Let $U \subset V \subset \widehat{H}$ be non-empty proper subsets with d_U and d_V being the associated Apollonian metrics. Let $\Delta = \sup_{u_1, u_2 \in U} d_V(u_1, u_2)$ be the diameter of the smaller subset within the larger. Then for every $x_1, x_2 \in U$:*

$$d_V(x_1, x_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_U(x_1, x_2). \quad (1.5)$$

If $\text{diam}_V(U) < +\infty$, the embedding $i : (U, d_V) \hookrightarrow (U, d_U)$ is a uniform contraction.

Proof: We will base our proof upon Birkhoff's inequality [Bir57] for cross-ratios on the projective real line. It is, in fact, a special case of our main theorem when $n = 1$. We will use it in the following version: Let $K = (a_1, a_2)$ be a non-empty open sub-interval of $J = (0, +\infty)$. The Hilbert distance of $s_1, s_2 \in K$ relative to K and J are given by:

$$d_K(s_1, s_2) = \left| \log[s_1, s_2; a_1, a_2] \right| \quad \text{and} \quad d_J(s_1, s_2) = \left| \log \frac{s_2}{s_1} \right|.$$

The quantity $\Delta = \text{diam}_J(K) = \log \frac{a_2}{a_1} \in (0, +\infty]$ measures the diameter of K for the J -metric. Birkhoff [Bir57, p.220] showed the fundamental inequality :

$$d_J(s_1, s_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_K(s_1, s_2), \quad \forall s_1, s_2 \in K. \quad (1.6)$$

Proof of (1.6): It suffices to show this for s_1 and s_2 infinitesimally close. So we differentiate with respect to s_2 at $s_2 = s_1 = s \in (a_1, a_2)$ and search for the optimal value of $\theta > 0$ so that for every $a_1 < s < a_2$: $\frac{1}{s} \leq \theta \frac{a_1 - a_2}{(s - a_1)(a_2 - s)}$, or equivalently

$$\theta \geq \inf_{a_1 < s < a_2} \frac{(s - a_1)(a_2 - s)}{s(a_2 - a_1)}. \quad (1.7)$$

The minimum value is at $s = \sqrt{a_1 a_2}$ and equals $\theta_{\min} = \frac{\sqrt{a_2} - \sqrt{a_1}}{\sqrt{a_2} + \sqrt{a_1}} = \tanh \frac{\log(a_2/a_1)}{4}$ which is therefore the desired contraction constant.

Now, returning to the general case let $x_1, x_2 \in U$ be distinct points. We have $d_V(x_1, x_2) \leq d_U(x_1, x_2)$ since the sup in the latter case is over a larger set. So we may assume that $\Delta = \text{diam}_V(U) < +\infty$ and also that $0 < d_V(x_1, x_2) \leq d_U(x_1, x_2) < +\infty$ (or else the statement is trivial). Let $\epsilon > 0$ and pick $v_1, v_2 \in V^c$ so that $d_V(x_1, x_2) \leq (1 + \epsilon) \log[x_1, x_2; v_1, v_2]$. To simplify calculations, we choose a transformation in $GM(\widehat{H})$ which maps v_1 to zero and v_2 to infinity. We recall that this preserves cross-ratios. By a slight abuse of notation we still write x_1, x_2 for the images in \widehat{H} of the corresponding points. We have then $0 < d_V(x_1, x_2) \leq (1 + \epsilon) \log \frac{\|x_2\|}{\|x_1\|}$ so in particular, $\|x_1\| < \|x_2\|$. When $u_1, u_2 \in U$ we have in these new coordinates, $\left| \log \frac{\|u_2\|}{\|u_1\|} \right| = |\log[u_1, u_2, 0, \infty]| \leq d_V(u_1, u_2) \leq \Delta < +\infty$. In other words, U is bounded away from the origin and infinity.

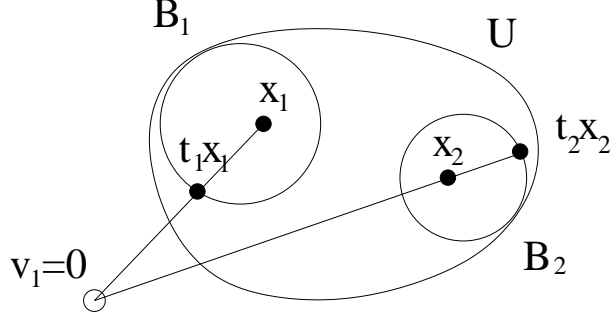


Figure 1: Construction of cross-ratios. $v_1 = 0$ and $v_2 = \infty$.

Consider now the formula for the distance of x_1, x_2 relative to U . It splits into a sum of two supremums (this splitting is one of the deeper reasons why the Apollonian metric is easy to handle):

$$d_U(x_1, x_2) = \sup_{u_1 \in U^c} \log \frac{\|x_2 - u_1\|}{\|x_1 - u_1\|} + \sup_{u_2 \in U^c} \log \frac{\|x_1 - u_2\|}{\|x_2 - u_2\|}.$$

The suprema of these two terms are denoted α_1 and α_2 . They are both finite. We define the Apollonian ball

$$B_1 = B_{\alpha_1}(x_1, x_2) = \left\{ u \in \widehat{H} : \frac{\|x_1 - u\|}{\|x_2 - u\|} < \alpha_1 \right\} \subset U$$

and similarly for the ball $B_2 = B_{\alpha_2}(x_2, x_1) \subset U$ (see Figure 1).

A priori B_1 is a generalized open ball containing x_1 but as U is bounded B_1 must be an open ball in the usual bounded sense (and α_1 must be greater than one). Now let $t_1 x_1$ (with $0 < t_1 < 1$) be the unique intersection of the segment $\{t x_1 : 0 \leq t \leq 1\}$ and the sphere $\partial B_{u_1}(x_1, x_2)$. Similarly, let $t_2 x_2$ (with $1 < t_2 < +\infty$) be the unique intersection between the segment $\{t x_2 : 1 \leq t \leq +\infty\}$ and $\partial B_{u_2}(x_2, x_1)$ (see Figure 1). Then $t_1 x_1, t_2 x_2 \in \text{Cl } U$ and $\|t_1 x_1\| < \|x_1\| < \|x_2\| < \|t_2 x_2\|$. From the way we defined t_1 and t_2 we have the following lower bound

$$\begin{aligned} d_U(x_1, x_2) = \alpha_1 + \alpha_2 &= \log \frac{\|x_2 - t_1 x_1\|}{\|x_1 - t_1 x_1\|} \times \frac{\|x_1 - t_2 x_2\|}{\|x_2 - t_2 x_2\|} \\ &= \log \frac{\|x_2 - t_1 x_1\|}{\|x_1\| - \|t_1 x_1\|} \times \frac{\|x_1 - t_2 x_2\|}{\|t_2 x_2\| - \|x_2\|} \\ &\geq \log \frac{\|x_2\| - \|t_1 x_1\|}{\|x_1\| - \|t_1 x_1\|} \times \frac{\|t_2 x_2\| - \|x_1\|}{\|t_2 x_2\| - \|x_2\|}. \end{aligned} \quad (1.8)$$

The last expression is the cross-ratio of the four (ordered) points on the positive real line $0 < \|t_1 x_1\| < \|x_1\| < \|x_2\| < \|t_2 x_2\| < +\infty$. Let us write $J = (0, \infty)$, $K = (\|t_1 x_1\|, \|t_2 x_2\|)$ and $s_1 = \|x_1\|$, $s_2 = \|x_2\|$. By our construction $\text{diam}_J(K) = \log \frac{\|t_2 x_2\|}{\|t_1 x_1\|} \leq d_V(t_1 x_1, t_2 x_2) \leq \text{diam}_V(U) = \Delta$, where we used that $t_1 x_1, t_2 x_2 \in \text{Cl } U$ and that $v_1 = 0, v_2 = \infty \in V$. Also $d_K(\|x_1\|, \|x_2\|) \leq d_U(x_1, x_2)$ by the above bound (1.8). So using Birkhoff's inequality (1.6) we

get

$$d_V(x_1, x_2)(1 + \epsilon)^{-1} \leq d_J(s_1, s_2) \leq \left(\tanh \frac{\text{diam}_J(K)}{4} \right) d_K(s_1, s_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_U(x_1, x_2),$$

and since $\epsilon > 0$ was arbitrary we see that

$$d_V(x_1, x_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_U(x_1, x_2),$$

which is what we aimed to show. \square

2 Some applications

In the one dimensional case, the result of Birkhoff [Bir57] has a vast variety of applications related to Perron-Frobenius type of results and the presence of spectral gaps of real operators contracting a real convex cone, see e.g. [Bal00]. In the case of complex operators similar spectral gap results were obtained first in [Rug10] and then simplified in [Dub09] using a complex Hilbert metric and the 2-dimensional version of the UCP for the Apollonian metric. We discuss in the following some possible applications in the case of arbitrary dimension.

Corollary 2.1 *Let $U \subset V$ and Δ be as in the Main theorem and write $\Gamma(V, U) = \{\gamma \in GM(\widehat{H}) : \gamma(V) \subset U\}$ for the elements of the conformal group that map V into U . Then for every $\gamma \in \Gamma(V, U)$ we have $\gamma^{-1} \in \Gamma(U^c, V^c)$ and the mappings $\gamma : (V, d_V) \rightarrow (V, d_V)$ and $\gamma^{-1} : (U^c, d_{U^c}) \rightarrow (U^c, d_{U^c})$ are $(\tanh \frac{\Delta}{4})$ -Lipschitz.*

Proof: $\gamma \in \Gamma(U, V)$ preserves cross-ratios, and $\gamma(V) \subset U$ so writing $\theta = \tanh \Delta/4$ we have for $v_1, v_2 \in V$:

$$d_V(\gamma(v_1), \gamma(v_2)) \leq \theta d_U(\gamma(v_1), \gamma(v_2)) \leq \theta d_{\gamma(V)}(\gamma(v_1), \gamma(v_2)) \leq \theta d_V(v_1, v_2).$$

The inverse map is bijective so it maps U^c into V^c . We have the same bound for its contraction rate since

$$\text{diam}_V(U) = \text{diam}_{U^c}(V^c) = \sup_{v_1, v_2 \in V} \sup_{u_1, u_2 \in U^c} \log[u_1, u_2; v_1, v_2]. \quad \square$$

Corollary 2.2 *In finite dimension when $\text{Cl } U \subset \text{Int } V$ for the topology of $(\widehat{H}, \widehat{d})$, then from compactness we see that $\text{diam}_V(U) < +\infty$ so the embedding $(U, d_U) \hookrightarrow (V, d_V)$ is a strict Lipschitz contraction.*

Lemma 2.3 *Suppose that $U \subset B(x_0, R)$, $R < \infty$. Then*

$$\|u_1 - u_2\| \leq \frac{R}{2} d_U(u_1, u_2), \quad \forall u_1, u_2 \in U. \quad (2.9)$$

Suppose that $U \subset V$ and that $r = \text{dist}(U, V^c) = \sup_{u \in U, w \in V^c} \|u - w\| > 0$. Then

$$d_V(u_1, u_2) \leq \frac{2}{r} \|u_1 - u_2\|, \quad \forall u_1, u_2 \in U. \quad (2.10)$$

Proof: When $x \in B(x_0, R)$ and h is small we get from a straight-forward calculation:

$$d_B(x, x+h) = \frac{2R}{R^2 - \|x - x_0\|^2} \|h\| + o(h).$$

Thus, $ds = \frac{2R}{R^2 - \|x - x_0\|^2} \|dx\| \geq \frac{2}{R} \|dx\|$ and $\|v_1 - v_2\| \leq d_B(v_1, v_2) \leq d_V(v_1, v_2)$ (since $V \subset B$).

When $B(u_1, r), B(u_2, r) \subset V$ then for $w \in V^c$: $\frac{\|u_2 - w\|}{\|u_1 - w\|} \leq 1 + \frac{\|u_2 - u_1\|}{r}$ and $d_V(u_1, u_2) \leq 2 \log \left(1 + \frac{\|u_2 - u_1\|}{r} \right) \leq \frac{2}{r} \|u_2 - u_1\|$. \square

Theorem 2.4 *Let $U \subset V$ be non-empty proper subsets of $(\widehat{H}, \widehat{d})$ such that $\text{Cl } V \neq \widehat{H}$ and $\Delta = \text{diam}_V(U) < +\infty$. Let $\gamma_1, \dots, \gamma_k \in \Gamma(V, U)$ and write*

$$\Lambda \equiv \Lambda(\gamma_1, \dots, \gamma_k) = \bigcap_{n \geq 1} \text{Cl} \bigcup_{1 \leq i_1, \dots, i_n \leq k} \gamma_{i_1} \circ \dots \circ \gamma_{i_n}(V)$$

for the associated limit set. Then Λ is compact and has Hausdorff and Box dimensions not greater than $-\log k / \log \tanh \frac{\Delta}{4}$.

Proof: Pick $q \in \widehat{H} \setminus \text{Cl } V$ and map q to infinity by an inversion in q . In the new coordinates V is bounded so by the previous Lemma, Hilbert distances are bounded by Apollonian distances. At level $n \geq 1$ each set in the finite union has diameter not greater than $r = \Delta(\tanh \frac{\Delta}{4})^{n-1}$ which becomes arbitrarily small as $n \rightarrow \infty$. There are $N_r = k^n$ elements in the union. As Λ is closed and has finite covers of arbitrarily small diameters it is compact and we have the bound

$$\dim_H(\Lambda) \leq \limsup_n \frac{\log N_r}{\log 1/r} = \frac{\log k}{\log \tanh \frac{\Delta}{4}}. \quad \square$$

When the images $\text{Cl}(\gamma_i(V))$, $1 \leq i \leq k$ are pairwise disjoint the Hausdorff dimension may also be obtained from a Bowen-like formula as in [Rug08] or [MU98]. We omit the details. Note that we do not assume here that H is finite dimensional.

Remark 2.5 *In finite dimension $d \geq 2$ the Apollonian metric for an open ball $V = B(0, R)$ is the same as the hyperbolic metric for the ball, i.e. $ds = 2r/(r^2 - \|x\|^2)\|dx\|$. In this case it is well-known that if γ maps V inside V and $\gamma(V)$ has bounded diameter then γ is a uniform contraction.*

Other metrics may be constructed from the Apollonian metric (cf. [Has04]). Let $v \in H^*$, $\|h\| \leq \|v\|/4$ and write $x = \frac{\langle h, v \rangle}{\langle v, v \rangle} \in [-1/4, 1/4]$ and $\|h\|^2 = \|v\|^2(x^2 + y^2)$. Calculus shows that $|\frac{1}{2} \log((1+x)^2 + y^2) - x| \leq x^2 + y^2$ (when $x \geq -1/4$). Therefore,

$$\left| \log \frac{\|v+h\|}{\|v\|} - \langle I(v), h \rangle \right| = \left| \log \frac{\|v+h\|}{\|v\|} - \frac{\langle v, h \rangle}{\langle v, v \rangle} \right| \leq \frac{\langle h, h \rangle}{\langle v, v \rangle}.$$

We assume in the following that U is open. Let $x \in U$ and set $r = \inf_{u \in U^c} d(x, u) > 0$. When $u_1, u_2 \in U^c$ and $\|h\| \leq r/4$ we get:

$$|\log[x, x+h; u_1, u_2] - \langle I(x - u_1) - I(x - u_2), h \rangle| \leq 2\|h\|^2/r^2.$$

It follows that the following limit exists and define a Finsler (pseudo-) norm on the tangent space of U :

$$p_{U,x}(h) \equiv \lim_{t \rightarrow 0} \frac{1}{t} d_U(x + th, x) = \sup_{u_1, u_2 \in U^c} |\langle I(x - u_1) - I(x - u_2), h \rangle|. \quad (2.11)$$

It is only a pseudo-norm when U^c is contained in a generalized ball, since in that case $p_{U,x}$ may vanish in some directions. If $\gamma : [0, 1] \rightarrow U$ is a continuous path then we may define its (pseudo-) length to be

$$\ell(\gamma) \equiv \limsup_{\delta \rightarrow 0} \sum_{k=0}^n d_U(\gamma(t_{k+1}), \gamma(t_k)),$$

where $0 = t_0 < t_1 < \dots < t_n = 1$ and $t_{k+1} - t_k < \delta$. Then

$$d_U^{\text{in}}(x, y) = \inf \{ \ell(\gamma) : \gamma \in C([0, 1], U), \gamma(0) = x, \gamma(1) = y \} \quad (2.12)$$

defines a (pseudo-)metric which in [Has04] was coined the Apollonian inner metric. When γ is peicewise C^1 we have $\ell(\gamma) = \int_0^1 p_{U,x}(\dot{\gamma}(t)) dt$. Another possiblity is to maximize (2.11) over directions. This leads to a conformal Riemannian metric $ds = g_U(x) \|dx\|$ with

$$g_U(x) = \sup_{\|h\|=1} p_{U,x}(h) = \sup_{u_1, u_2 \in U^c} \frac{\|u_1 - u_2\|}{\|x - u_1\| \|x - u_2\|}. \quad (2.13)$$

An advantage of this metric is perhaps that it distinguishes points when U^c contains at least two points. It is easy to see that $g_U(x)$ is continuous (as we assumed U to be open). We write $d_U^{\text{Rie}}(x, y)$ for the Riemannian distance of x and y with respect to this metric.

Corollary 2.6 *Let $U \subset V \subset \widehat{H}$ (with $\text{Cl } V \neq \widehat{H}$) be non-empty proper subsets and let $\Delta = \sup_{u_1, u_2 \in U} d_V(u_1, u_2)$ be the diameter of the smaller subset within the larger with respect to the Apollonian metric. Then for every $x, y \in U$:*

$$p_{V,x}(h) \leq \left(\tanh \frac{\Delta}{4} \right) p_{U,x}(h), \quad h \in E, \quad (2.14)$$

$$d_V^{\text{in}}(x, y) \leq \left(\tanh \frac{\Delta}{4} \right) d_U^{\text{in}}(x, y), \quad (2.15)$$

$$d_V^{\text{Rie}}(x, y) \leq \left(\tanh \frac{\Delta}{4} \right) d_U^{\text{Rie}}(x, y). \quad (2.16)$$

Proof: For $x, x+th \in U$ we have by the Main Theorem $\frac{1}{t} d_V(x, x+th) \leq \tanh \frac{\Delta}{4} \frac{1}{t} d_U(x, x+th)$. The first inequality follows. The second follows by taking limits in the right order. For the Riemannian metric one has

$$g_V(x) \leq \sup_{\|h\|=1} p_{V,x}(h) \leq \sup_{\|h\|=1} \left(\tanh \frac{\Delta}{4} \right) p_{U,x}(h) = \left(\tanh \frac{\Delta}{4} \right) g_U(x)$$

which yields the last inequality. \square

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